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Abstract

Schrödinger investigated entanglement in two-particle state vectors by assuming measurement, finding out if the nearby particle is in a given state vector ψ_1 or not. Without interaction with the distant particle, just on account of the entanglement, the distant particle is steered into a certain state vector. In Schrödinger's finite-dimensional case thus any distant-particle state vector can be reached. This theory was extended to infinite-dimensional spaces by the author. The present paper completes the extension by throwing light on the fine structure of steering.

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1. Introduction

When in 1935 Einstein *et al* launched their revolutionary EPR paradox [1], many deep-thinking foundationally-minded physicists followed suit. Among them were Furry [2] and Schrödinger [3, 4]. The latter author introduced the now widely used concept of entanglement, but also that of disentanglement and of steering or distant steering [5]. Schrödinger's approach and indignation can be seen in his words [3] p 556: "It is rather discomforting that the theory should allow a system to be steered or piloted into one or the other type of state at the experimenter's mercy in spite of his having no access to it". This is made even more clear in his next paper [4], p 446: "... in general a sophisticated experimenter can, by a suitable device which does not involve measuring non-commuting variables, produce a non-vanishing probability of driving the system into any state he chooses". He had two-particle pure states with non-singular reduced density operators, and finite-dimensional state spaces of particles in mind. (This will be obvious after the detailed study in this paper.)

Distant steering in the case of two-particle state vectors that have reduced density operators with infinite-dimensional ranges was studied by the present author [6, 7]. The present paper is actually a completion of the former study with insight into the fine structure of steering. Wiseman *et al* extended steering to mixed two-particle states [8].

This study focusses on *two-particle state vectors* that have *infinite-dimensional ranges* of reduced density operators. (The theory is general, but the fine structure studied does not show up in the trivial, finite-dimensional case.)

2. The distant state

It is well known that one of the basic quantum-mechanical relations is the so-called *trace rule*, which expresses the probability $p(P, \rho)$ of occurrence of a quantum event (projector) P in a quantum state (density operator) ρ by the simple formula $p(P, \rho) = \mathrm{tr}(P\rho)$. 'Occurrence' is defined by measurement, but it is an astonishing quantum-mechanical fact that this notion has a 'two-dimensional multitude':

(i) One can take any observable (Hermitian operator) A of which P is an eigenprojector corresponding to an (arbitrary) eigenvalue a, i.e., an operator the spectral form of which is $A = aP + P^\perp A P^\perp$ where $P^\perp \equiv 1 - P$ (and the second term does not have the eigenvalue a). If in the measurement of A the result a is obtained, then one says that P has occurred.

(ii) The observable specified in (i) can be measured in whatever way: ideally (the textbook case), when the Lüders formula [9] gives the change of state, in more general non-demolition measurement (older synonyms: repeatable measurement, or measurement of the first kind), in measurement in which the result is not preserved (non-repeatable or second-kind measurement).

It is not widely known that if one has a bipartite system in any correlated state (density operator) ρ_{12}, i.e., when $\rho_{12} \neq \rho_1 \otimes \rho_2$, where the tensor factors are the reduced density operators, then, if a first-subsystem event P_1 occurs (in the sense defined in the preceding passages with the two multitudes of varieties), then the second subsystem *ipso facto*, i.e., without any interaction between the measuring instrument and the subsystem, makes transition from the subsystem state (reduced density operator) $\rho_2 \equiv \mathrm{tr}_1 \rho_{12}$ to the following state (density operator) in $\mathcal{H}_2$:

$$p^{-1} \mathrm{tr}_1(\rho_{12} P_1), \tag{1a}$$

where

$$p \equiv \mathrm{tr}_{12}(P_1 \rho_{12}) \tag{1b}$$

is the probability of the occurrence of P_1 in the state ρ_{12}. We write under the partial trace P_1 instead of $P_1 \otimes I_2$, where I_2 is the identity operator in $\mathcal{H}_2$. (A proof of (1a) and (1b) is given in [10], subsection 6.B.)

We need two steps of confining ourselves to special cases from relation (1a) to reach Schrödinger's steering. We want to do this in the antilinear representation of bipartite state vectors (vectors of norm one) [7]. (The indices 1, 2 and 12 show in which space the entity is.)

3. The antilinear representation

There is an isomorphism from the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$, where the factors are complex separable (finite or countably infinite-dimensional) Hilbert spaces (state spaces of the subsystems) to *antilinear Hilbert–Schmidt operators* A_a that map $\mathcal{H}_1$ into $\mathcal{H}_2$, determined by the *partial scalar product*

$$\forall \, |\Phi\rangle_{12} \quad \rightarrow \quad A_a : \tag{2a}$$

$$\forall \, |\psi\rangle_1 : \quad (A_a |\psi\rangle_1)_2 \equiv \langle\psi|_1|\Phi\rangle_{12}, \tag{2b}$$

where $\langle\psi|_1|\Phi\rangle_{12}$ is the *partial scalar product* over subsystem 1.

Each antilinear operator A_a defined by (2b) determines its adjoint $A_a^\dagger$, which maps $\mathcal{H}_2$ into $\mathcal{H}_1$. The adjoint is determined via the relation

$$\forall |\psi\rangle_1, |\phi\rangle_2 : \quad (A_a|\psi\rangle_1, |\phi\rangle_2)_2 = \left(|\psi\rangle_1, A_a^\dagger|\phi\rangle_2\right)_1^*, \tag{3}$$

where the brackets stand for scalar products, which are antilinear in the first factor, and the asterisk denotes complex conjugation.

Relation (2b) also implies $\operatorname{tr}\left(A_a^\dagger A_a\right) < \infty$. This is the relation that makes the antilinear operators A_a and $A_a^\dagger$ Hilbert–Schmidt ones.

4. Two steps of special cases

Now we take the first step toward taking a special case of (1a) and (1b). By P_{χ_i} we denote the projector onto the one-dimensional subspace spanned by the unit vector χ_i, and $i = 1, 2, 12$ keeps track of the Hilbert space to which the entity belongs (even if it is superfluous, it is useful for transparency). Naturally, $P_{\chi_i}\psi_i = \chi_i(\chi_i, \psi_i)_i$, $i = 1, 2, 12$.

Theorem 1. *If one has any bipartite state vector Φ_{12}, i.e., $\rho_{12} \equiv P_{\Phi_{12}}$, and one goes over to the antilinear representation A_a of Φ_{12}, the occurrence of any first-subsystem event P_1 brings about the following second-subsystem state:*

$$p^{-1}\left[A_a P_1 A_a^\dagger\right]_2, \tag{4a}$$

where

$$p \equiv (\Phi_{12}, P_1 \Phi_{12})_{12} \tag{4b}$$

is the probability of the event P_1 in the state Φ_{12}.

Proof. As to the antilinear representation, we are going to utilize (2b) and (3), and the fact that numbers undergo complex conjugation when taken to the left from an antilinear operator. Let $\{\phi_2^n : \forall n\}$ be a complete orthonormal basis in $\mathcal{H}_2$, and let $\{\psi_1^k : \forall k\}$ be a complete orthonormal basis in $\overline{\mathcal{R}(P_1)}$, the topological closure of the range of P_1: $P_1 = \sum_k P_{\psi_1^k}$. Then

$$\left(\phi_2^n, \left[\operatorname{tr}_1\left(P_{\Phi_{12}} P_1\right)\right]_2 \phi_2^{n'}\right)_2 = \sum_k \left(\psi_1^k \phi_2^n, P_{\Phi_{12}} \psi_1^k \phi_2^{n'}\right)_{12} = \sum_k \left(\psi_1^k \phi_2^n, \Phi_{12}(\Phi_{12}, \psi_1^k \phi_2^{n'})_{12}\right)_{12}$$

$$= \sum_k \left(\psi_1^k \phi_2^{n'}, \Phi_{12}\right)_{12}^* \left(\phi_2^n, A_a \psi_1^k\right)_2$$

$$= \sum_k \left(\phi_2^{n'}, A_a \psi_1^k\right)_2^* \left(\phi_2^n, A_a \psi_1^k\right)_2 = \sum_k \left(A_a^\dagger \phi_2^{n'}, \psi_1^k\right)_1 \left(\phi_2^n, A_a \psi_1^k\right)_2$$

$$= \sum_k \left(\psi_1^k, A_a^\dagger \phi_2^{n'}\right)_1^* \left(\phi_2^n, A_a \psi_1^k\right)_2 = \left(\phi_2^n, \left[A_a \sum_k P_{\psi_1^k}\left[A_a^\dagger \phi_2^{n'}\right]_1\right]_2\right)_2$$

$$= \left(\phi_2^n, \left[A_a P_1 \left[A_a^\dagger \phi_2^{n'}\right]_1\right]_2\right)_2 = \left(\phi_2^n, \left[A_a P_1 A_a^\dagger\right]_2 \phi_2^{n'}\right)_2. \qquad \square$$

The second step toward taking a special case of (4a), (4b) is confining ourselves to *elementary* first-subsystem *events* (ray projectors) $P_1 \equiv P_{\psi_1}$, where ψ_1 is an arbitrary state vector.

Theorem 2. *If an elementary event P_{ψ_1} occurs on the first subsystem in a state vector Φ_{12} (or A_a), then the second subsystem finds itself in the state described by the state vector*

$$A_a \psi_1 / \|A_a \psi_1\|, \tag{5a}$$

and the probability of the occurrence of P_{ψ_1} is

$$p = \|A_a \psi_1\|^2. \tag{5b}$$

Proof. As to the claimed probability (5b), from (4b) one has

$$p = \text{tr}_{12}\big(P_{\Phi_{12}} P_{\psi_1}\big).$$

Introducing complete orthonormal bases $\big\{\psi_1^k : \forall k; \psi_1^{k=1} \equiv \psi_1\big\}$ and $\big\{\phi_2^n : \forall n\big\}$, one obtains

$$p = \sum_n \big(\psi_1 \phi_2^n, P_{\Phi_{12}} \psi_1 \phi_2^n\big)_{12}.$$

Applying the projector, and taking out one scalar product from the other, one further has

$$p = \sum_n \big(\Phi_{12}, \psi_1 \phi_2^n\big)_{12}\big(\psi_1 \phi_2^n, \Phi_{12}\big)_{12} = \sum_n \big(\psi_1 \phi_2^n, \Phi_{12}\big)_{12}^*\big(\phi_2^n, A_a \psi_1\big)_2$$

$$= \sum_n \big(\phi_2^n, A_a \psi_1\big)_2^*\big(\phi_2^n, A_a \psi_1\big)_2 = \|A_a \psi_1\|^2.$$

To derive claim (5a), we start with (4a), and utilize the above basis in $\mathcal{H}_2$:

$$\begin{aligned}
\big(\phi_2^n, p^{-1}\big[A_a P_{\psi_1} A_a^\dagger\big]_2 \phi_2^{n'}\big)_2 &= \big(\phi_2^n, \big[p^{-1/2} A_a \psi_1\big]_2\big(\psi_1, [p^{-1/2} A_a^\dagger \phi_2^{n'}]_1\big)_1\big)_2 \\
&= \big(\psi_1, \big[p^{-1/2} A_a^\dagger \phi_2^{n'}\big]_1\big)_1^*\big(\phi_2^n, \big[p^{-1/2} A_a \psi_1\big]_2\big)_2 \\
&= \big(\big[p^{-1/2} A_a \psi_1\big]_2, \phi_2^{n'}\big)_2\big(\phi_2^n, \big[p^{-1/2} A_a \psi_1\big]_2\big)_2 \\
&= \big(\phi_2^n, \big[p^{-1/2} A_a \psi_1\big]_2\big([p^{-1/2} A_a \psi_1]_2, \phi_2^{n'}\big)_2\big) \\
&= \big(\phi_2^n, P_{[p^{-1/2} A_a \psi_1]_2} \phi_2^{n'}\big)_2. \qquad \square
\end{aligned}$$

One should note that after the second equality the scalar product is complex conjugated because before the first equality, A_a is seen to act after the projector, hence also on the numbers that come out as a result of the projection. In contrast, if we read the next to last expression one step backward, the scalar product is extracted without complex conjugation though it is to the right of A_a. The reason is that A_a acts in $\mathcal{H}_1$, and the scalar product (a number) appears in $\mathcal{H}_2$ after the action of A_a. *Schrödinger's steering* is defined for the occurrence of an arbitrary elementary first-subsystem event P_{ψ_1} in an arbitrary bipartite state vector Φ_{12}. As is shown in theorem 2, this boils down to *mapping $\mathcal{H}_1$ into $\mathcal{H}_2$ by A_a* (the antilinear representative of Φ_{12}).

5. Polar factorization

To bring out the full power of the antilinear representation, one should perform the two polar factorizations of A_a [11]:

$$A_a = U_a \rho_1^{1/2}, \tag{6a}$$

$$A_a = \rho_2^{1/2} U_a Q_1, \tag{6b}$$

where $\rho_i \equiv \text{tr}_j P_{\Phi_{12}}, i, j = 1, 2, i \neq j$, is the ith subsystem state (reduced density operator), and U_a is an antilinear unitary operator mapping the topologically closed range $\overline{\mathcal{R}(\rho_1)}$ onto the topologically closed range $\overline{\mathcal{R}(\rho_2)}$ (these subspaces are always equally dimensional), and, finally, Q_1 is the range projector of ρ_1.

The operator U_a is called *the correlation operator*. It is the only precise mathematical entity expressing the quantum correlations inherent in a bipartite state (known to the author).

Remark 1. As is seen in (6b), Schrödinger's steering maps $\mathcal{H}_1$ into $\mathcal{R}\left(\rho_2^{1/2}\right)$. Actually, it is a surjection, i.e., an 'onto' map [6]. This is, of course, non-trivial only in the case of infinite-dimensional ranges (of ρ_i, $i = 1, 2$), when one should have in mind the known proper inclusion relations

$$\mathcal{R}(\rho) \subset \mathcal{R}(\rho^{1/2}) \subset \overline{\mathcal{R}(\rho)} \tag{7}$$

valid for any density operator with infinite-dimensional range.

6. The largest probability of steering

We proceed by analyzing (5a), (5b) to gain detailed insight into Schrödinger's steering.

Theorem 3. *(A) Two first-subsystem state vectors ψ_1 and ψ_1' give, upon measurement, the same steering in subsystem 2 if and only if*

(i) the range projections are positively collinear

$$Q_1\psi_1 = cQ_1\psi_1', \qquad c > 0, \tag{8a}$$

or equivalently

(ii) if they determine by projection the same state vector in $\overline{\mathcal{R}(\rho_1)}$:

$$Q_1\psi_1/\|Q_1\psi_1\| = Q_1\psi_1'/\|Q_1\psi_1'\|. \tag{8b}$$

(B) Of all elementary events P_{ψ_1} in $\mathcal{H}_1$ that give one and the same state vector in $\mathcal{H}_2$ by steering the largest probability of occurrence has the one that lies entirely in $\overline{\mathcal{R}(\rho_1)}$, or, equivalently, the component of which in the null space of A_a is zero.

Proof.   (A) *Sufficiency.*   Let (8a) be valid.   Since $\rho_1 = \rho_1 Q_1$, and $\rho_1^{1/2} = \rho_1^{1/2}Q_1$, (6$a$) implies $A_a = A_a Q_1$. Hence, applying A_a to (8a), one obtains $A_a\psi_1 = cA_a\psi_1'$, and $\|A_a\psi_1\| = c\|A_a\psi_1'\|$. Finally,

$$A_a\psi_1/\|A_a\psi_1\| = A_a\psi_1'/\|A_a\psi_1'\|. \tag{9}$$

Necessity. If relation (9) is valid, then

$$A_a(\psi_1/\|A_a\psi_1\| - \psi_1'/\|A_a\psi_1'\|) = 0 = Q_1(\psi_1/\|A_a\psi_1\| - \psi_1'/\|A_a\psi_1'\|)$$

(A_a and Q_1 have the same null space). Finally,

$$Q_1\psi_1 = (\|A_a\psi_1\|/\|A_a\psi_1'\|)Q_1\psi_1'.$$

Thus, (8a) is satisfied.

Clearly, (8b) implies (8a). Conversely, (8a) gives $\|Q_1\psi_1\| = c\|Q_1\psi_1'\|$. Relation (8$b$) ensues from (8$a$) and this relation.

(B) Relation (5b) implies

$$p = \|A_a\psi_1\|^2 = \|A_a Q_1\psi_1\|^2 = (\|Q_1\psi_1\|^2)[\|A_a(Q_1\psi_1/\|Q_1\psi_1\|)\|^2]. \tag{10}$$

All vectors specified in (8b) have the second factor after the last equality in (10) in common. Therefore, the probability is largest when the first factor (after the last equality in (10)) is largest, i.e., when it is one. $\qquad\square$

7. The fine structure of infinite-dimensional ranges

In this section, we make a deviation from our two-particle study to one Hilbert space and a given density operator with an infinite-dimensional range in it.

Remark 2. Let ρ be a density operator with an infinite-dimensional range. Writing '$\oplus$' for the union of disjoint sets, and '$\ominus$' when set-theoretically subtracting a subset from a larger set, the proper-inclusion chain (7) implies

$$\overline{\mathcal{R}(\rho)} = \mathcal{R}(\rho) \oplus (\mathcal{R}(\rho^{1/2}) \ominus \mathcal{R}(\rho)) \oplus (\overline{\mathcal{R}(\rho)} \ominus \mathcal{R}(\rho^{1/2})). \tag{11}$$

Lemma 1. *Let $\{\psi_k : \forall k\}$ be a complete orthonormal eigenbasis of ρ in $\overline{\mathcal{R}(\rho)}$, and let $\{r_k : \forall k\}$ be the corresponding positive spectrum of ρ (with possible repetitions of equal eigenvalues in general). Let, further, $\overline{\mathcal{R}(\rho)} \ni \psi = \sum_k a_k \psi_k$, with $\forall k: a_k \in \mathbf{C}$ be an arbitrary element, i.e., $\sum_k |a_k|^2 < \infty$. Then*

$$\psi \in \mathcal{R}(\rho) \quad \Leftrightarrow \quad \sum_k \left| r_k^{-1} a_k \right|^2 < \infty, \tag{12}$$

and

$$\psi \in \mathcal{R}(\rho^{1/2}) \quad \Leftrightarrow \quad \sum_k \left| r_k^{-1/2} a_k \right|^2 < \infty. \tag{13}$$

Proof. $\{\Leftarrow \text{ in}(12)\}$. Assuming the validity of the second expression in (12), we define $\phi \equiv \sum_k r_k^{-1} a_k \psi_k$. Then one has $\psi = \rho\phi$, i.e., the first expression in (12) holds true. $\{\Rightarrow \text{ in}(12)\}$. If ψ belongs to the range, there exists $\phi = \sum_k b_k \psi_k$, $\sum_k |b_k|^2 < \infty$ and $\rho\phi = \psi$. Since $\forall k: a_k = r_k b_k$, one has $\forall k: \sum_k \left| r_k^{-1} a_k \right|^2 < \infty$.

Equivalence (13) is proved analogously.

$\square$

Lemma 2. *The square root $\rho^{1/2}$ of any density operator ρ with an infinite-dimensional range maps in a one-to-one way $\overline{\mathcal{R}(\rho)}$ onto $\mathcal{R}(\rho^{1/2})$, and by this it maps $\mathcal{R}(\rho^{1/2})$ onto $\mathcal{R}(\rho)$, and $\{\overline{\mathcal{R}(\rho)} \ominus \mathcal{R}(\rho^{1/2})\}$ onto $\{\mathcal{R}(\rho^{1/2}) \ominus \mathcal{R}(\rho)\}$, i.e., (14a)–(14c) is valid*

$$\mathcal{R}(\rho^{1/2}) \oplus \{\overline{\mathcal{R}(\rho)} \ominus \mathcal{R}(\rho^{1/2})\} = \overline{\mathcal{R}(\rho)} \tag{14a}$$

$$\downarrow \qquad\qquad \downarrow \tag{14b}$$

$$\mathcal{R}(\rho) \oplus \{\mathcal{R}(\rho^{1/2}) \ominus \mathcal{R}(\rho)\} = \mathcal{R}(\rho^{1/2}). \tag{14c}$$

Proof. That $\rho^{1/2}$ maps $\overline{\mathcal{R}(\rho)}$ into $\mathcal{R}(\rho^{1/2})$ is obvious from (6b). To prove that it is an 'onto' map, let $\psi = \sum_k a_k \psi_1$ (cf lemma 1) be an arbitrary element of the latter range. Then, according to (13), also $\phi \equiv \sum_k r_k^{-1/2} a_k \psi_k$ is an element of $\overline{\mathcal{R}(\rho)}$. Applying $\rho^{1/2}$ to it, we obtain ψ. Assuming *ab contrario* that $\phi, \phi' \in \overline{\mathcal{R}(\rho)}$, $\phi \neq \phi'$ and $\rho^{1/2}\phi = \rho^{1/2}\phi'$, one arrives at $\rho^{1/2}(\phi - \phi') = 0$, i.e., a non-zero element is taken into zero. This is not possible because $\rho^{1/2}$ has the same null space as ρ, and it is the orthocomplement of $\overline{\mathcal{R}(\rho)}$.

The first arrow in (14b), i.e., the map $\rho^{1/2}$ that it denotes, is obvious in the 'into' sense because $\rho^{1/2}\rho^{1/2} = \rho$. Let $\psi = \sum_k a_k \psi_k$ be an arbitrary element of $\mathcal{R}(\rho)$. Then, according to (12), $\sum_k \left| r_k^{-1} a_k \right|^2 < \infty$. Then also $\sum_k \left| r_k^{-1/2} a_k \right|^2 < \infty$ (compare the first inclusion in (7) with (12) and (13)). Hence, we can define $\phi \equiv \sum_k r_k^{-1/2} a_k$, and we have $\rho^{1/2}\phi = \psi$. Thus, we are dealing with an 'onto' map.

Finally, the last claim is an immediate consequence of the preceding two, as easily seen.

$\square$

8. Back to steering

It was shown in previous work [11] that the correlation operator does not just map $\overline{\mathcal{R}(\rho_1)}$ onto $\overline{\mathcal{R}(\rho_2)}$. It takes by similarity transformation the positive part of the one reduced density operator into that of the other

$$\rho_2 = U_a \rho_1 U_a^{-1} Q_2, \tag{15}$$

where Q_2 is the range projector of ρ_2.

Lemma 3. *The correlation operator preserves decomposition (11):*

$$U_a \overline{\mathcal{R}(\rho_1)} = \overline{\mathcal{R}(\rho_2)}, \tag{16a}$$

$$U_a \mathcal{R}(\rho_1) = \mathcal{R}(\rho_2), \tag{16b}$$

$$U_a \left(\mathcal{R}\left(\rho_1^{1/2}\right) \ominus \mathcal{R}(\rho_1) \right) = \left(\mathcal{R}\left(\rho_2^{1/2}\right) \ominus \mathcal{R}(\rho_2) \right), \tag{16c}$$

$$U_a \left(\overline{\mathcal{R}(\rho_1)} \ominus \mathcal{R}\left(\rho_1^{1/2}\right) \right) = \left(\overline{\mathcal{R}(\rho_2)} \ominus \mathcal{R}\left(\rho_2^{1/2}\right) \right). \tag{16d}$$

Proof. In lemma 1 we made the choice $\forall k: \ \rho_1 \psi_1^k = r_k \psi_1^k$. Applying the correlation operator, one obtains $\forall k: \ \left(U_a \rho_1 U_a^{-1} \right) \left(U_a \psi_1^k \right) = r_k \left(U_a \psi_1^k \right)$. Defining $\forall k: \ \psi_2^k \equiv U_a \psi_1^k$, one can, account of (15), write $\forall k: \ \rho_2 \psi_2^k = r_k \psi_2^k$. Since $\mathcal{R}(\rho_i) \ i = 1, 2$ is the linear manifold spanned by the eigenvectors $\{\psi_i^k : \forall k\}, \ i = 1, 2$, (16b) is valid.

According to (13), $\sum_k \left| r_k^{-1/2} a_k \right|^2 < \infty$ is satisfied for every element $\psi_1 = \sum_k a_k \psi_1^k$ that belongs to $\mathcal{R}\left(\rho_1^{1/2}\right)$. Applying U_a, one has $(U_a \psi_1) = \sum_k a_k^* \psi_2^k$, and $\sum_k \left| r_k^{-1/2} a_k^* \right|^2 = \sum_k \left| r_k^{-1/2} a_k \right|^2 < \infty$. Thus, $U_a \mathcal{R}\left(\rho_1^{1/2}\right) = \mathcal{R}\left(\rho_2^{1/2}\right)$. The rest in the claim is evident. $\qquad \square$

Theorem 4. *The antilinear representative A_a of a given bipartite state vector Φ_{12} (cf (2a,b)) that implies reduced density operators with infinite-dimensional ranges maps in a one-to-one way $\overline{\mathcal{R}(\rho_1)}$ onto $\mathcal{R}\left(\rho_2^{1/2}\right)$, and by this it maps $\mathcal{R}\left(\rho_1^{1/2}\right)$ onto $\mathcal{R}(\rho_2)$, and $\left\{ \overline{\mathcal{R}(\rho_1)} \ominus \mathcal{R}\left(\rho_1^{1/2}\right) \right\}$ onto $\left\{ \mathcal{R}\left(\rho_2^{1/2}\right) \ominus \mathcal{R}(\rho_2) \right\}$. This is made more transparent by the following relations:*

$$\mathcal{R}\left(\rho_1^{1/2}\right) \oplus \left\{ \overline{\mathcal{R}(\rho_1)} \ominus \mathcal{R}\left(\rho_1^{1/2}\right) \right\} = \overline{\mathcal{R}(\rho_1)} \tag{17a}$$

$$\downarrow \qquad\qquad\qquad \downarrow \tag{17b}$$

$$\mathcal{R}(\rho_2) \ \oplus \ \left\{ \mathcal{R}\left(\rho_2^{1/2}\right) \ominus \mathcal{R}(\rho_2) \right\} = \mathcal{R}\left(\rho_2^{1/2}\right). \tag{17c}$$

Proof. The claim of the theorem is evident having in mind the polar factorization (6b) of A_a, lemma 3 and lemma 2 in application to ρ_2. $\qquad \square$

9. Conclusion

In the formalism the map A_a that represents antilinearly any given bipartite state vector Φ_{12} *actually performs the Schrödinger steering.* If the composite-system state vector implies infinite-dimensional reduced density operators $\rho_i, \ i = 1, 2$, then the mapping has a fine structure:

  (i) If the measurement of ψ_1 is that of an eigenvector of ρ_1 corresponding to a positive eigenvalue, then actually the corresponding eigenvector $\psi_2 = U_a \psi_1$ is *distantly measured*. This simplest case was extensively studied in [11, 12] in the non-selective version of measurement, when all results are taken into account in contrast to Schrödinger's steering, in which the selective version of measurement is considered with only one result—that of obtaining 1 for P_{ψ_1}.

(ii) All other elements of $\mathcal{R}(\rho_2)$ can be obtained by steering that results from direct measurement of a vector ψ_1 from $\mathcal{R}\left(\rho_1^{1/2}\right)$. This case was studied in detail, again in the non-selective version of measurement, in [7].

(iii) Finally, the elements of $\left(\mathcal{R}\left(\rho_2^{1/2}\right) \ominus \mathcal{R}(\rho_2)\right)$ can be reached by steering when direct measurement of vectors from $\left(\overline{\mathcal{R}(\rho_1)} \ominus \mathcal{R}\left(\rho_1^{1/2}\right)\right)$ is performed.

Besides, the vectors from the range of ρ_1 give, by selective measurement, the largest probability. Hence, the null space of ρ_1 is best discarded in steering.

The paradoxical physical meaning of distant steering is not discussed in this paper. Quantum-mechanical insight into the nice EPR-type entanglement experiments of Scully *et al* [13, 14] (a thought and a real experiment) gained by the present author recently [15, 16] has led to the conclusion that distant correlations are paradoxical only in the Einsteinian absolute-property interpretation of quantum mechanics . If one takes resort to the alternative, the relative-property interpretation, a kind of Everettian approach, then nothing is paradoxical.

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